Theorem Of The Keplerian Kinematics

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Abstract: *We state a theorem of kinematics that explains the Keplerian motion as suggested by the works of the literature. We prove it as true and then demonstrate some of its mathematical consequences among which the Newton's and Einstein's postulates must be reviewed. This work embeds no postulate at all but only some trivial kinematics calculations.*

1 Introduction

It has been largely reported in the literature[2-9] that the velocity of a Keplerian orbiter is the addition of a translation velocity and a rotation velocity, especially by using the hodograph plane representation of the motion. This kinematics property has always been presented as a consequence of the Newton's gravitational postulate of attraction. It appears however that such a peculiar geometric property could stand alone and be prior to the Newton's acceleration which can then be deduced from the kinematics. The purpose of the present work is to investigate this perspective.

We will first state a kinematics theorem and then prove that it forecasts the three laws of Kepler. After doing so we will look at some consequences among which the gravitation appears as causing the rotation, but not the attraction, and a gravitational acceleration can not be equivalent to a mechanical acceleration.

2 Theorem

Accordingly to the works of the literature it is possible to state the following theorem :

Theorem : *The velocity of a Keplerian orbiter is the addition of an uniform rotation velocity and an uniform translation velocity, both coplanar.* (1)

Let us demonstrate that this theorem leads to the three laws of Kepler, with no need at all to consider that the Newton's acceleration would be the cause of these kinematics. At the contrary we will demonstrate that it is a consequence.

3 Kepler's laws

Mathematically the above theorem can be written as so :

$$\mathbf{v} = \mathbf{v_R} + \mathbf{v_T}$$

with

$$\mathbf{v_R} = \boldsymbol{\omega} \times \mathbf{r} \quad \text{and} \quad v_R = \|\mathbf{v_R}\| = \omega\, r = \text{constant}$$
$$\mathbf{v_T} = \mathbf{constant} \tag{2}$$

In this expression $\boldsymbol{\omega}$ is the frequency of rotation, $\mathbf{r}$ is the vector radius, $\mathbf{v_R}$ is the uniform rotation velocity and $\mathbf{v_T}$ the uniform translation velocity. Note that here the index R does not stand for "radial" but for "rotation", and the index T does not stand for "tangential" but for "translation" (see figure 1).

The first consequence of the above expression is the validity of the following one :

$$\dot{\boldsymbol{\omega}}\, r = -\dot{r}\, \boldsymbol{\omega} \tag{3}$$

From the relations (2) and (3) we can calculate the acceleration which is the derivative of the velocity with respect to time :

$$\mathbf{a} = \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \mathbf{v} = -\frac{\boldsymbol{\omega}}{r^2} \times \left(\mathbf{r} \times \left(\mathbf{r} \times \mathbf{v}\right)\right) \tag{4}$$

Defining the massless angular momentum like R.H. Battin[9] did as

$$\mathbf{L} = \mathbf{r} \times \mathbf{v} \tag{5}$$

the final expression of the acceleration is given by :

$$\mathbf{a} = -\frac{L\, v_R}{r^3}\, \mathbf{r} \tag{6}$$

Therefore the acceleration and the vector radius are colinear and this forces the angular momentum to be a constant, as awaited for a central field motion :

$$\mathbf{L} = \mathbf{constant} \tag{7}$$

Now from this we observe that the vector product of the rotation velocity with the angular momentum leads trivially to :

$$\mathbf{v_R} \times \mathbf{L} = v_R^2 \left(1 - \frac{\mathbf{v_R} \cdot \mathbf{v_T}}{v_R^2}\right) \mathbf{r} \tag{8}$$

The scalar version of this equation is therefore :

$$p = \left(1 + e\cos\theta\right) r \quad \text{with} \quad p = \frac{L}{v_R} \quad \text{and} \quad e = \frac{v_T}{v_R} \tag{9}$$

This is the equation of a conic where p is the semi latus rectum, e is the eccentricity and θ is the true anomaly, i.e. the angle between $\mathbf{v_T}$ and $\mathbf{v_R}$ which is also the angle between the direction of the periapsis and the vector radius. This is the expression of the Kepler's first law.

Note that the vector expression of the eccentricity is given by :

$$\mathbf{e} = \frac{\mathbf{v_T} \times \mathbf{L}}{L\ v_R} \tag{10}$$

Therefore the translation velocity is always perpendicular to the main axis of the conic, which direction is the one of the vector eccentricity. The figure 1 exhibits both the rotation and the translation velocities at different positions on a conic.

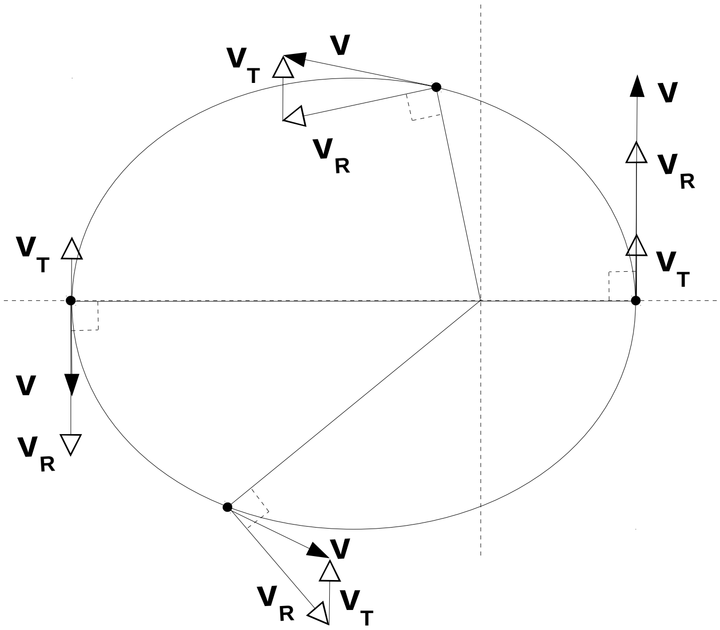

Figure 1 : *representation of the velocity* $\mathbf{v}$ *of a Keplerian orbiter made of the addition of a uniform translation velocity* $\mathbf{v_T}$ *and a uniform rotation velocity* $\mathbf{v_R}$ *at four different positions over a Keplerian conic. Note that* $\mathbf{v_T}$ *is always perpendicular to the main axis of the conic while* $\mathbf{v_R}$ *is always perpendicular to the vector radius.*

Let us now notice that the scalar multiplication of the total velocity and the vector radius leads to :

$$\mathbf{r}\ .\ \mathbf{v} = \mathbf{r}\ .\ \mathbf{v_T} = r\,\dot{r} \quad \text{thus} \quad \dot{r} = v_T \sin\theta \tag{11}$$

Using this last expression it is trivial to show that the angular momentum can be presented as the multiplication of the square of the vector radius and the derivative of the true anomaly with respect to time :

$$L = r^2\,\dot{\theta} \tag{12}$$

This last expression is very well known, being described for instance by L. Landau and E. Lifchitz in their course “Mechanics”[1]. It shows that the areal velocity, defined as $f = r^2\,\dot{\theta}/2$, must be a constant as far as the angular momentum also is. Therefore the expression (12) is nothing else but the second law of Kepler.

Note that the time derivative of the true anomaly $\dot{\theta}$ and the frequency of rotation ω are related by the following formula :

$$\dot{\theta} = \omega \left(1 + e \cos \theta\right) = \omega \frac{p}{r} \quad \text{or} \quad r \dot{\theta} = p \omega \tag{13}$$

Now integrating the expression (12) over a complete period of revolution for an ellipse, as described by L. Landau and E. Lifchitz, and knowing that L and v_R are two constants, we are trivially led to the following formula :

$$L\, v_R = 4 \pi^2 \frac{a^3}{T^2} = k = \text{constante} \tag{14}$$

This is the expression of the third law of Kepler.

We have therefore demonstrated that the theorem (1) forecasts indeed the three laws of Kepler.

4 Consequences

4.1 Newton's postulate

The acceleration of a Keplerian orbiter is given by the expression (6) which is exactly the one of Newton at the condition that :

$$L\, v_R = GM \tag{15}$$

where G is the universal constant of gravitation and M the mass of the central body causing the motion of the orbiter. We also note that the same condition is required to make the relation (14) of the third law of Kepler compatible with the one of Newton.

From the kinematics point of view Newton has therefore implicitly postulated that $L\, v_R = GM$. As far as the kinematics does not take any physical property into account, as the mass for instance, such a postulate has indeed to be setup in order to connect the kinematics with the physics.

However the minus sign of the expression (6) is not relative to an attraction but to a rotation because the acceleration is centripetal. This result is consistent with the experiment that shows undoubtedly that the astral bodies are rotating around each others instead of collapsing by attraction.

4.2 Galileo's principle of equivalence

The theorem (1) is mass independent and therefore it shows that a motion in a gravitational field is mass independent as shown by Galileo.

4.3 Mechanical energy

Calculating the square of the expression (2) it is trivial to define a kinematic energy, i.e. a massless energy as follows :

$$E_M = \frac{1}{2} v^2 - \frac{L\, v_R}{r} = \frac{1}{2} v_R^2 \left(e^2 - 1\right) \quad (16)$$

Multiplying this last expression by the mass of the orbiter, and considering the formula (15), we get directly the usual expression of the mechanical energy as described in classical mechanics[1].

4.4 Body falling

If we hold an object in the hand, its velocity is null but it must nonetheless respect the theorem (1). This leads to consider that the rotation and the translation velocities must be of the same amplitude but opposite directions :

$$\mathbf{v_R} = -\mathbf{v_T} \quad (17)$$

The rotation velocity is provided by the gravitation while the translation velocity is relative to the constraints that disable the orbitation.

Now if we let fall the object to the floor we slightly decrease the constraints applied to the object which enters a conic motion where the amplitude of $\mathbf{v_T}$ is slightly lower than the one of $\mathbf{v_R}$. The eccentricity of the conic, given by the expression (9), is therefore close but lower to 1 : $e = v_T / v_R \approx 0.999\ldots$

Such a conic with such an eccentricity is a very sharp ellipse as presented on the figure 2. Locally the object looks like falling on a straight line but in reality it falls on a conic.

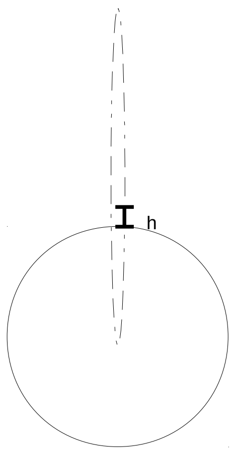

Figure 2 : *fall of an object from a height h, at the surface of a planet. Locally the fall looks like a straight line but it is not, this is the part of a conic.*

What we describe here is the Einstein's thought experiment of an observer into an elevator[11]. The observer will be able to know if he is at the surface of a planet rather than mechanically thrusted. In the first case the object will fall on a conic, in the second one the object will fall on a straight line. The kinematics therefore disagree with the Einstein's equivalence principle.

4.5 Mechanical versus gravitational accelerations

Let us consider an orbiter on a perfect circular orbit, so having $\mathbf{v_T = 0}$. Its acceleration is of course given by the expression (6). Let us now apply a mechanical force **F** provided by an engine, the total acceleration will then become :

$$\mathbf{a} = -\frac{L\,v_R}{r^3}\,\mathbf{r} + \frac{\mathbf{F}}{m} \tag{18}$$

where m is the mass of the orbiter. Integrating this expression must lead to the expression (2) of the velocity because the orbiter must respect the theorem (1). We shall therefore verify :

$$\mathbf{v_R} = \int -\frac{L\,v_R}{r^3}\,\mathbf{r}\,dt \quad \text{and} \quad \mathbf{v_T} = \int \frac{\mathbf{F}}{m}\,dt \tag{19}$$

We see here that the mechanical acceleration can only provide the translation while the gravitational one provides the rotation. This is quite logical as far as a force must have a connection to the axis of rotation to cause a rotation, but the mechanical force has no connection to the axis. At the contrary the force of gravitation has a connection to it, this is the gravitation itself, so it can cause a rotation.

This explains why it is impossible to accelerate mechanically a satellite by keeping it on a circular orbit. Any thrust, as short and local as it could be, whatever its direction, will cause $\mathbf{v_T}$ to be not null any more, and therefore the eccentricity $e = v_T / v_R$ to be different from zero, so the circular orbit will change into an ellipse.

The conclusion is that the gravitational and the mechanical accelerations are of two different natures, the first one causing the rotation while the second can only cause a translation. Here again this result of the kinematics disagrees with the Einstein's equivalence principle[11].

4.6 Rotation of the galaxies

Vera Rubin has shown that the stars inside the disks of the galaxies have a velocity incompatible with the Newton's theory of the gravitation[10]. The figure 3 gives a typical example of what is expected from the Newton's postulate and what is actually measured.

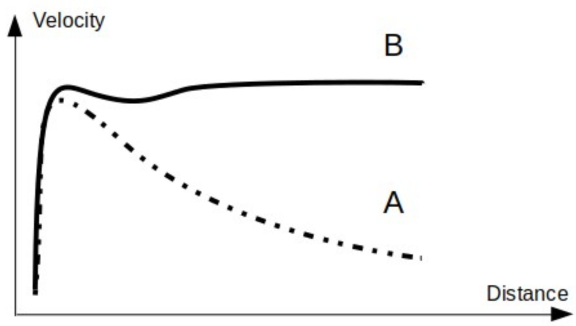

Figure 3 : Typical v*elocity of the stars in a galactic disk with respect to their distance to the center of the galaxy. The doted curve A is the one expected with the theory of Newton, the plain curve B is what is actually measured.*

At a first approximation we can consider that the stars in the galactic disk have a circular orbit and therefore their velocity is given by the third law of Kepler (14) : $v = \sqrt{k/r}$.

For Newton the numerator k = GM = constant, and consequently the velocity must decrease when the distance r increases.

For the kinematics $k = L\, v_R = L\, \omega\, r$, therefore $v = \sqrt{L\, \omega}$ and the velocity can remain constant whatever the distance, at the condition that $L\, \omega$ also is. But $L\, \omega$ has the dimension of a massless energy, consequently if the stars of the galactic disk are populating the same massless energy level $E = L\, \omega$, they will have the same velocity independently of their distance to the center, and the curve B of the figure 3 can be explained.

The kinematics can therefore explain the experimental measures without dark matter, but considering that the galaxies are structured around some energy levels that are mathematically analogous to a macroscopic version of the Planck-Einstein relation.

5 Conclusion

In this work we have analyzed the kinematics of the Keplerian motion without using any hypothesis, nor opinion, nor postulate. We have proven that the Kepler's laws can be easily deduced from a simple theorem of kinematics, suggested by numerous works of the literature.

A theorem is not a postulate, the first being provable while the second is not. The theorem (1) is a geometrical truth and therefore can not be ignored by the scientist, like any theorem. Would we imagine to ignore the Pythagora's theorem for some convenience of any kind ? Of course not. The same goes for the theorem (1).

Such a theorem of kinematics is however not sufficient to make a complete theory of the gravitation, because it can not take into account the physical properties of the systems, as the mass for instance. This is only a geometric description.

Nonetheless this description makes unmissable some mandatory consequences among which the gravitation causes the rotation but not the attraction, the Newton's postulate reduces to $L\, v_R = GM$, the Einstein's equivalence principle can not be correct, the rotation of the galactic disks could be explained without dark matter.

The Einstein's equivalence principle being to be reviewed, does this mean that the whole General Relativity could be wrong ? Not at all in our opinion, because we know the excellent agreement of the GR with the experimental measures. So it does only mean that the GR could be based on an other foundation than this Einstein's postulate, and doing so it should enforce this theory. For instance we may think about an extension of the GR at other scales than the only astronomic one. Indeed the Einstein's constant $8\, \pi\, G / c^4$ of the GR is using the universal constant G coming from the Newton's constant GM. This is fine for the astronomic studies but we know that it is not applicable to the atomic scale where the Newton's force is replaced by the Coulomb's one, having the same mathematical structure but using an other constant ($e^2/4\pi\varepsilon_0$). We do not know how to relate the Newton's and the Coulomb's constants, but the kinematics suggests that Lv_R could replace them in both cases. This might be an interesting track to follow.

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