Theorem of the Keplerian Velocity

Herve Le Cornec*
(Dated: February 28, 2021)

Based on numerous works of the literature we state a theorem of kinematics describing the Keplerian motion by the means of its velocity. We first prove that this theorem forecasts the three laws of Kepler and then we study some of its mathematical consequences. Among them the Newton's gravitational acceleration is forecasted, however it is centripetal but not attractive, and it can not be equivalent to a mechanical acceleration. Additionally a possible solution to the problem of the rotation of the galaxies emerges.

I. INTRODUCTION

The Newton's gravitational acceleration is considered as the reason for the Keplerian motion. However as far as an acceleration can not exist without a velocity, we may wonder if it exists a typical gravitational velocity for the Keplerian motion, as it exists the typical gravitational acceleration of Newton. Many works of the literature, mainly using the hodograph plane representation of the motion[1–8], report some information about the Keplerian velocity : it is the addition of a uniform rotation velocity and a uniform translation velocity. Such a simple and peculiar mathematical structure of the velocity interrogates, so we investigated it.

We found that a very simple theorem of kinematics regarding the velocity leads to the forecast of the three laws of Kepler, as well as the mathematical structure of the Newton's acceleration. This is what we demonstrate in the first part of this article.

In the second part we study some of the consequences of this theorem. Among them the kinematic interpretation of the acceleration is somehow different from the one of Newton, even if its mathematical structure is confirmed. A kinematic constant takes the place of the famous numerator GM of Newton, and this has some drastic consequences on all the theories that uses the GM factor, as the General Relativity or the existence of dark matter.

II. STATEMENT OF THE THEOREM

According to the works of the literature we state the following theorem :

Theorem 1 *The velocity of a Keplerian orbiter on a fixed orbit is always the sum of a uniform rotation velocity and a uniform translation velocity, both coplanar.*

Let us demonstrate that any orbiter which velocity is given by this theorem will have a motion respecting the three laws of Kepler.

* herve.le.cornec@free.fr

III. PROOF OF THE THEOREM

Mathematically the above theorem can be written as so :

$$\begin{aligned}&\vec{v} = \vec{v}_R + \vec{v}_T\\&\text{with}\\&v_R = \|\vec{v}_R\| = \|\vec{\omega}\times\vec{r}\| = \omega r = constant\\&v_T = \|\vec{v}_T\| = constant\end{aligned} \tag{1}$$

In this expression $\vec{\omega}$ is the frequency of rotation, $\vec{r}$ is the vector radius, $\vec{v}_R$ is the uniform rotation velocity and $\vec{v}_T$ the uniform translation velocity. Note that here the index R does not stand for "radial" but for "rotation", and the index T does not stand for "tangential" but for "translation" (see figure 1). From the above relationship we get a trivial but important expression :

$$\dot{\omega} r + \omega \dot{r} = 0 \tag{2}$$

From the relations 1 and 2 we can calculate the acceleration which is the derivative of the velocity with respect to time :

$$\vec{a} = \dot{\vec{\omega}}\times\vec{r} + \vec{\omega}\times\vec{v} = -\frac{\vec{\omega}}{r^2}\times(\vec{r}\times(\vec{r}\times\vec{v})) \tag{3}$$

Defining the "massless angular momentum" like R.H. Battin [9] did as

$$\vec{L} = \vec{r}\times\vec{v} \tag{4}$$

the final expression of the acceleration is given by :

$$\vec{a} = -\frac{L\, v_R}{r^3}\vec{r} \tag{5}$$

Therefore the acceleration and the vector radius are colinear and this forces the angular momentum to be a constant, as awaited for a central field motion :

$$\vec{L} = \vec{constant} \tag{6}$$

Now from this we observe that the vector product of the rotation velocity with the angular momentum leads trivially to :

$$\vec{v}_R\times\vec{L} = v_R^2\left(1 + \frac{\vec{v}_R.\vec{v}_T}{v_R^2}\right)\vec{r} \tag{7}$$

Therefore the scalar version of this expression is

$$p = (1 + e\cos\theta)\, r \quad \text{with} \quad p = \frac{L}{v_R} \quad \text{and} \quad e = \frac{v_T}{v_R} \tag{8}$$

This is the equation of a conic where p is the semi latus rectum, e is the eccentricity and θ is the true anomaly, i.e. the angle between $\vec{v}_R$ and $\vec{v}_T$ which is also the angle between the direction of the periapsis and the vector radius. This is the expression of the Kepler's first law.

Note that the vector expression of the eccentricity is given by :

$$\vec{e} = \frac{\vec{v}_T \times \vec{L}}{L v_R} \tag{9}$$

and that explains why the translation velocity is always perpendicular to the main axis of the conic.

The figure 1 exhibits both the rotation and the translation velocities at different positions on a conic.

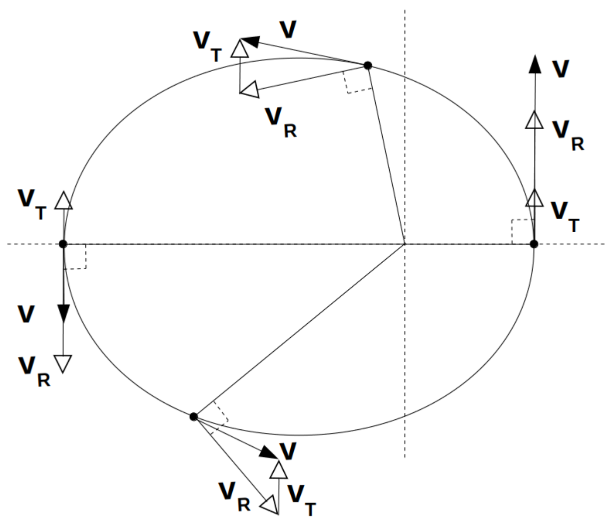

FIG. 1. The velocity of a Keplerian orbiter $\vec{v}$ on a fixed orbit is always the sum of a uniform rotation velocity $\vec{v}_R$, perpendicular to the vector radius, and a uniform translation velocity $\vec{v}_T$, which direction is always perpendicular to the main axis of the conic. Both are coplanar and have a constant norm all along the trajectory.

Let us now note that the scalar multiplication of the total velocity and the vector radius leads to :

$$\vec{r}.\vec{v} = \vec{r}.\vec{v}_T = r\dot{r} \quad \text{thus} \quad \dot{r} = v_T sin\theta \tag{10}$$

Using this last expression it is trivial to show that the angular momentum can be presented as the multiplication of the square of the vector radius and the derivative of the true anomaly with respect to time :

$$L = r^2\dot{\theta} \tag{11}$$

This last expression is very well known, being described for instance by L. Landau and E. Lifchitz in their course "Mechanics" [10]. It shows that the areal velocity, defined as $f = r^2\dot{\theta}/2$, must be a constant as far as the angular momentum also is. Therefore the expression 11 is nothing else but the second law of Kepler.

Note that the time derivative of the true anomaly $\dot{\theta}$ and the frequency of rotation ω are related by the following formula :

$$\dot{\theta} = \omega(1 + ecos\theta) = \omega\frac{p}{r} \quad \text{or} \quad r\dot{\theta} = p\omega \tag{12}$$

Now integrating the expression 11 over a complete period T of revolution for an ellipse, as described by L. Landau and E. Lifchitz, and knowing that L and v_R are two constants, we are trivially led to the following formula :

$$L v_R = 4\pi^2\frac{a^3}{T^2} = k = constant \tag{13}$$

This is the expression of the third law of Kepler.

We have therefore demonstrated that the theorem 1 forecasts indeed the three laws of Kepler. This result is not a point of view, neither an hypothesis, neither a postulate, but this is a geometric reality that we can not ignore.

IV. SOME CONSEQUENCES

A. Newton's gravitational postulate

The acceleration of a Keplerian orbiter is given by the expression 5 which is exactly the one of Newton at the condition that :

$$L v_R = GM \tag{14}$$

where G is the universal constant of gravitation and M the mass of the central body causing the motion of the orbiter. We also note that the same condition is required to make the relation 13 of the third law of Kepler compatible with the one of Newton.

From the kinematics point of view Newton has therefore implicitly postulated that $L v_R = GM$. Apart from that the mathematical structure of the acceleration that he proposed is fully compatible with the one required by the kinematics. However his interpretation of the sign of the acceleration disagrees with the one of the kinematics. Indeed for Newton the direction of the acceleration being opposed to the one of the vector radius, it is attractive, while for the kinematics it is centripetal. For the kinematics the gravitation does not cause the attraction but the rotation.

B. Galileo's principle of equivalence

The theorem 1 is mass independent and therefore it agrees that any motion in a gravitational field is mass independent, as shown by Galileo.

C. Mechanical energy

Calculating the square of the expression 1 it is trivial to define a kinematic energy, i.e. a massless energy as follows :

$$E_M = \frac{1}{2}v^2 - \frac{L\,v_R}{r} = \frac{1}{2}v_R^2(e^2-1) \qquad (15)$$

Multiplying this last expression by the mass of the orbiter, we get directly the usual expression of the mechanical energy as described in classical mechanics [10].

D. Body falling

If we hold an object in the hand, its velocity is null but it must nonetheless respect the theorem 1. This leads to consider that the rotation and the translation velocities must be of the same amplitude but opposite directions: $\vec{v}_R = -\vec{v}_T$. The rotation velocity is provided by the gravitation while the translation velocity is relative to the constraints that disable the orbitation.

Now if we let the object fall to the floor we slightly decrease the constraints applied to it, so we slightly decrase v_T, and the overall velocity is not null any more. The trajectory of the object is of course a conic with an eccentricity, given by the expression 8, close but lower to 1 : $e = v_T/v_R = 0.999...$ Such a conic with such an eccentricity is a very sharp ellipse as presented on the figure 2. Locally the object looks like falling on a straight line but in reality it falls on a conic. An observer confined into a tidy blind cabin, droping an object to the floor, will know if he is at the surface of a planet rather than mechanically thrusted. In the first case the object will fall on a conic ($v_R \neq 0$), while in the second case the object will fall on a stright line ($v_R = 0$).

E. Mechanical versus gravitational acceleration

Let us consider an orbiter of mass m on a perfect circular orbit, so having $\vec{v}_T = \vec{0}$. Its acceleration is of course given by the expression 5. Let us now apply a mechanical force $\vec{F}$ provided by an engine, the total acceleration will then become :

$$\vec{a} = -\frac{Lv_R}{r^3}\vec{r} + \frac{\vec{F}}{m} \qquad (16)$$

Integrating this expression must lead to the velocity defined by the expression 1 because the orbiter must respect the theorem 1. We shall therefore verify :

$$\vec{v}_R = \int_{t_0}^{t} -\frac{Lv_R}{r^3}\vec{r}dt \quad \text{and} \quad \vec{v}_T = \int_{t_0}^{t} \frac{\vec{F}}{m}dt \qquad (17)$$

We see here that the mechanical acceleration can only provide the translation while the gravitational one provides the rotation. This is quite logical as far as a force must have a connection to the axis of rotation to cause a rotation, but the mechanical force has no connection to the axis. At the contrary the force of gravitation has a connection to it, this is the gravitation itself, so it can cause a rotation.

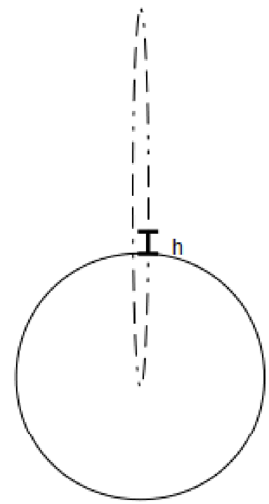

FIG. 2. Fall of an object from a height h, at the surface of a planet. Locally the fall looks like a straight line but it is not, this is the part of a conic.

This explains why it is impossible to accelerate mechanically a satellite by keeping it on a circular orbit. Any thrust, as short and local as it could be, whatever its direction, will cause $\vec{v}_T$ to be not null any more, and therefore the eccentricity $e = v_T/v_R$ to be different from zero, so the circular orbit will change into an ellipse.

The conclusion is that the gravitational and the mechanical accelerations are of two different natures, the first one causing the rotation while the second can only cause a translation.

F. Rotation of the galaxies

Vera Rubin has shown that the stars inside the disks of the galaxies have a velocity incompatible with the Newton's theory of the gravitation [11]. The figure 3 gives a typical example of what is expected from the Newton's postulate and what is actually measured. At a first approximation we can consider that the stars in the galactic disk have a circular orbit and therefore their velocity is given by the third law of Kepler 13 : $v = \sqrt{k/r}$.

For Newton the numerator $k = GM = constant$, and consequently the velocity must decrease when the distance r increases.

For the kinematics $k = Lv_R = L\omega r$, therefore $v = \sqrt{L\omega}$ and the velocity can remain constant whatever the distance, at the condition that $L\omega$ is constant too. But $L\omega$ has the dimension of a massless energy, consequently if the stars of the galactic disk are populating the same massless energy level, they will have the same velocity independently of their distance to the center of the galaxy, and the curve B of the figure 3 can be explained.

The kinematics can therefore explain the experimental measures, at the condition that the galaxies are structured around some energy levels that are mathematically analogous to a macroscopic version of the Planck-Einstein relation.

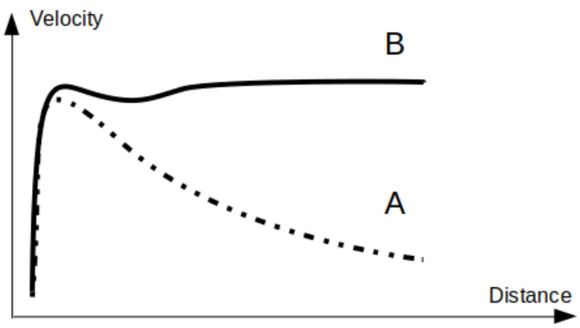

FIG. 3. Typical velocity of the stars in a galactic disk with respect to their distance to the center of the galaxy. The doted curve A is the one expected with the theory of Newton, the plain curve B is what is actually measured.

V. DISCUSSION

The theorem 1 is not an opinion, nor an hypothesis, nor a postulate, but a geometric reality that can not be ignored. As far as any physical theory of the gravitation must forecast the existence of the three laws of Kepler, it must also be consistent with this theorem. This is the case for the mathematical structure of the Newton's acceleration, but not for its interpretation, on two important points.

First, for the kinematics the direction of the acceleration is indeed opposed to the direction of the vector radius, like Newton said, however the acceleration is not attractive but centripetal. We should not speak of the "universal attraction", but of the "universal rotation". Looking at the astral bodies in our telescopes, we must admit that they are indeed rotating around each others rather than collapsing by attraction like two magnets would do.

Second, to be consistent with the kinematics the constant numerator GM of the Newton's acceleration must be equivalent to the constant numerator Lv_R of the kinematics (see equation 14). Implicitly Newton has then postulated that $GM = Lv_R$, and this postulate is working indeed pretty well at the scale of the solar system, as it has been fully experimented. However we know that the Newton's acceleration fails to explain the experimental observations at the galactic and atomic scales. This could be due to the factor GM that is drastically and universally constant, while Lv_R is more mathematically flexible and could vary at different scales. Let us indeed remember that the Coulomb's acceleration has the same mathematical structure as the kinematic acceleration 5, and therefore we are led to wonder if the theorem 1 could also be at work, Lv_R being then equal to the Coulomb's constant. This has to be investigated.

The Einstein's General Relativity (GR) also uses the constant GM as a key factor, but we know that it does not work at an atomic scale. It might then be interesting to investigate what changes could happen by using Lv_R instead. Would it then be possible to extend the GR at other scales ? At this regard we may remind that in the atomic model of Rutherford, the electron is rotating around the proton like a planet around the sun [14]. It was then opposed to him that a rotating electron has a centripetal acceleration and therefore, being charged, it must emit some radiations, making the atom unstable. However this argument was unfair because if the electron is really like a planet, it is in weightlessness. It feels no acceleration, then emits nothing, and its orbit is perpetually stable. But so far the only way that we have to explain the weightlessness is the GR, by the curvature of the space-time. Therefore we are led to wonder if the GR could be extended to the atomic scale by using Lv_R instead of GM. This has also to be investigated.

An other important consequence of the theorem is that a gravitational acceleration can not be equivalent to a mechanical acceleration, the first one causing a rotation while the second can only cause a translation (see sections IV D and IV E). This contradicts the Einstein's equivalence principle, at least as he exposed in his publications of 1907 and 1911 [12, 13]. Therefore, knowing that many people take this postulate as the corner stone of the GR, we may wonder if the GR is still correct. For us, no doubt on its validity because the predictions of the GR forecast a big lot of experimental observations, from the precession of Mercury to the existence of the black holes, passing through the gravitational waves and lenses, and much more. So what rather suggests the kinematics is only that the GR could be based on something else than the equivalence principle.

Last but not least, the theorem can explain the rotation of the galaxies in a very simple and straight forward manner. All the stars having the same massless energy level $E_0 = L\omega = constant$ (energy divided by the mass of the orbiter) will have the same velocity regardless to their distance to the center of the galaxy (see IV F). Because the theorem is purely geometric but embeds no physical concept, we can not figure out what would be the physical reason for such a structure, we can only certify that this is a possible mathematical solution to explain the rotation of the galaxies, with no need of any postulate like the existence of a strange dark matter or a magical acceleration appearing at long range. We can note however that $E_0 = L\omega$ is a macroscopic version of the Plank-Einstein relation $E = \hbar\omega$, and therefore the stars in the galactic disk are populating a constant energy level like electrons do in an atom or a molecule. Our current vision of the physics of the galaxies should then be reviewed.

To conclude we must remind that the kinematics alone can not replace a physics theory, because it can not consider any physical property of a system, like the mass for instance. The theorem 1 is only a geometric description of the motion. However all physical theories of the gravitation must at least respect the kinematics, therefore this theorem. This last then appears like an help to improve our physical theories, and why not extending their validity at other scales than their current ones.

[1] Orbit information derived from its hodograph, J. B. Eades, Tech. Rep. TM X-63301, NASA (1968)

[2] W. R. Hamilton, The hodograph, or a new method of expressing in symbolic language the Newtonian law of attraction, Proc. R. Ir. Acad. III , 344353 (1845).

[3] H. Abelson, A. diSessa and L. Rudolph, Velocity space and the geometry of planetary orbits, Am. J. Phys. 43 , 579-589 (1975).

[4] A. Gonzalez-Villanueva, H. N. Nunez-Yepez, and A. L. Salas-Brito, In veolcity space the Kepler orbits are circular, Eur. J. Phys. 17 , 168-171 (1996).

[5] T. A. Apostolatos, Hodograph: A useful geometrical tool for solving some difficult prob- lems in dynamics, Am. J. Phys. 71 , 261-266 (2003).

[6] E. I. Butikov, The velocity hodograph for an arbitrary keplerian motion, Eur. J. Phys. 21 (2000) 1-10

[7] D. Derbes, Reinventing the wheel: Hodographic solutions to the Kepler problems, Am. J. Phys. 69 , 481-489 (2001).

[8] R. H. Battin, An Introduction to the Mathematics and Methods of Astrodynamics, Revised Edition, American Institute of Aeronautics and Astronautics, Inc., Reston, 1999, §3.5, 126

[9] R. H. Battin, An Introduction to the Mathematics and Methods of Astrodynamics, Revised Edition, American Institute of Aeronautics and Astronautics, Inc., Reston, 1999, §3.3, 115

[10] L.Landau, E. Lifchitz, Mechanics, Ed. Mir, Moscow, 1966, §15

[11] V. Rubin, A Century of Galaxy Spectroscopy, The Astrophysical Journal, 451: 419ff (1995)

[12] A. Einstein, 1907, Über das Relativitätsprinzip und die aus demselben gezogenen Folgerungen, Jahrbuch der Radioaktivität und Elektronik, 4, 411462, 5, 98-99 (Berichtigungen).

[13] A. Einstein, 1911, Über den Einfluss der Schwerkraft auf die Ausbreitung des Lichtes. Annalen der Physik, 35, 898/908.

[14] E. Rutherford, "The Scattering of α and β Particles by Matter and the Structure of the Atom" [archive], Philosophical Magazine. Series 6, vol. 21. May 1911