Hamilton's kinematics to extend Newton's gravitational acceleration

Hervé Le Cornec*

Abstract: We present a simple but new kinematic analysis of the Keplerian velocity that W.R Hamilton described by the means of the hodograph representation. This analysis expectedly predicts the Kepler's laws as well as the mathematical structure of Newton's gravitational acceleration. However its interpretation of the body falling is different from the current one, and it considers Newton's physical factor GM as a particular case of a broader kinematic factor, free of any physical constraint. Using this factor instead of GM, to agree the kinematics, could enable to extend Newton's gravitational acceleration to other scales, as suggested by two examples of applications.

Introduction

In 1845 W.R. Hamilton demonstrated[1] by the way of the hodograph representation that the velocity of all Keplerian orbiters is simply the addition of two uniform velocities, one of rotation plus one of translation. This geometric fact has been confirmed since by many authors in the literature[2-8]. Consequently the derivation of this velocity leads to a centripetal acceleration, but not to an attractive one, like Newton assumes, although no author noted this obvious mathematical fact.

The trajectory of a mobile experiencing an attractive acceleration is collinear to the acceleration, while the trajectory of a mobile experiencing a centripetal acceleration is perpendicular to the acceleration. The centripetal and attractive accelerations are thus of very different natures. They cannot be assimilated nor confused.

Therefore, there is here an inconsistency between the geometry of the Keplerian motion and Newton's postulate. We then decided to make a deeper kinematic analysis of the Keplerian motion, starting from Hamilton's velocity, in order to understand more about this inconsistency.

Like Hamilton, the authors used mainly the hodograph representation, but this way of doing things makes the kinematics of the motion difficult to handle mathematically. We present here a simpler and straight forward kinematic analysis of this simple Keplerian velocity, that has not yet been described in these terms in the literature, at our knowledge.

As expected, this analysis shows that the three laws of Kepler as well as the mathematical structure of Newton's acceleration derive from the Keplerian velocity demonstrated by Hamilton. It shows that Newton's attractive acceleration for the planetary motions is kinematically fully compatible with a centripetal interpretation. However, this is not true any more for the body falling, for which the kinematics predicts a conic trajectory while Newton's assumes an accelerated straight line. There is thus a conflict that we will propose to solve.

* herve.le.cornec@free.fr

The kinematic analysis also demonstrates that the physical factor GM[10] of Newton's gravitational acceleration must correspond to the kinematic factor Lv_R (L being the norm of the massless angular momentum, and v_R being Hamilton's rotation velocity). GM and Lv_R have the same dimension (m^3s^{-2}), and same role (numerator), in the mathematical expression of the Keplerian acceleration. This equivalence leads us to propose the extension of the scope of Newton's acceleration at other scales than the only astronomic one.

Kinematic analysis

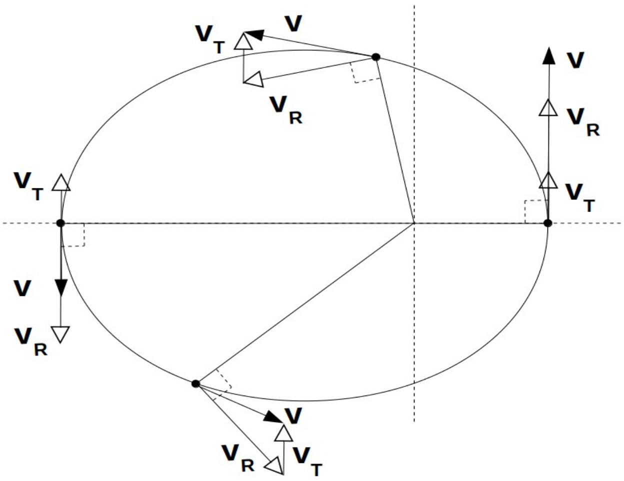

Figure 1 : the Keplerian velocity **v** *decomposed into its rotation velocity* $\mathbf{v_R}$ *and its translation velocity* $\mathbf{v_T}$.

The velocity **v** of any Keplerian orbiter has been widely described in the literature, it is simply the vector addition of two uniform velocities, one of rotation, $\mathbf{v_R}$, plus one of translation, $\mathbf{v_T}$. Its simplest mathematical expression is then as follows:

$$\mathbf{v} = \mathbf{v_R} + \mathbf{v_T}$$

with

$$\begin{aligned} &\mathbf{v_R} = \boldsymbol{\omega} \times \mathbf{r} \\ &v_R = \|\mathbf{v_R}\| = \omega\, r = \text{constant} \\ &\mathbf{v_T} = \textbf{constant} \end{aligned} \tag{1}$$

In this expression $\boldsymbol{\omega}$ is the vector frequency of rotation, perpendicular to the plane of the orbit, and **r** is the vector radius, from the focus of the orbit to the orbiter. Note that $\mathbf{v_T}$ and $\mathbf{v_R}$ are coplanar all along the orbit. Take care, in this expression the index R means “rotation” but not “radial”, while the index T stands for “translation” but not “tangential”. The figure 1 shows these two velocities on a typical Keplerian orbit.

Now let us demonstrate that coming from this definition of the orbital velocity we can predict the existence of Kepler’s laws as well as Newton’s acceleration, or at least its mathematical structure.

The first consequence of the above expression (1) is the validity of the following one by derivation of v_R with respect to time ($\boldsymbol{\omega}$ and $\dot{\boldsymbol{\omega}}$ being colinear) :

$$\dot{\boldsymbol{\omega}}\, r = -\dot{r}\, \boldsymbol{\omega} \qquad (2)$$

From the relations (1) and (2) we can then calculate the acceleration which is the derivative of the velocity with respect to time :

$$\mathbf{a} = \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \mathbf{v} = -\frac{\boldsymbol{\omega}}{r^2} \times \left(\mathbf{r} \times \left(\mathbf{r} \times \mathbf{v}\right)\right) \qquad (3)$$

Now defining the massless angular momentum like R.H. Battin[8] did as

$$\mathbf{L} = \mathbf{r} \times \mathbf{v} \qquad (4)$$

the final expression of the acceleration is given by :

$$\mathbf{a} = -\frac{L\, v_R}{r^3}\, \mathbf{r} \qquad (5)$$

Therefore the acceleration and the vector radius are colinear and this forces the angular momentum to be a constant, as awaited for a central field motion :

$$\mathbf{L} = \mathbf{constant} \qquad (6)$$

Note that the expression (5) of the acceleration has the same mathematical structure as Newton’s gravitational acceleration, but it is of course centripetal.

Now from this we observe that the vector product of the rotation velocity with the angular momentum leads trivially to :

$$\mathbf{v_R} \times \mathbf{L} = v_R^2 \left(1 + \frac{\mathbf{v_R} \cdot \mathbf{v_T}}{v_R^2}\right) \mathbf{r} \qquad (7)$$

The scalar version of this equation is therefore :

$$p = \left(1 + e \cos\theta\right) r \quad \text{with} \quad p = \frac{L}{v_R} \quad \text{and} \quad e = \frac{v_T}{v_R} \qquad (8)$$

This is the equation of a conic where p is the semi latus rectum, e is the eccentricity and θ is the true anomaly, i.e. the angle between $\mathbf{v_T}$ and $\mathbf{v_R}$ which is also the angle between the direction of the perigee and the vector radius. This is the expression of Kepler’s first law.

Note that the eccentricity vector is given by :

$$\mathbf{e} = \frac{\mathbf{v_T} \times \mathbf{L}}{L\, v_R} \qquad (9)$$

Therefore the translation velocity is always perpendicular to the main axis of the conic, which direction is the one of the vector eccentricity. The figure 1 exhibits both the rotation and the translation velocities at different positions on a conic.

Let us now notice that the scalar multiplication of the total velocity and the vector radius leads to :

$$\mathbf{r} \cdot \mathbf{v} = \mathbf{r} \cdot \mathbf{v_T} = r\, \dot{r} \quad \text{thus} \quad \dot{r} = v_T \sin\theta \qquad (10)$$

Using this last expression it is trivial to show that the angular momentum can be presented as the multiplication of the square of the vector radius and the derivative of the true anomaly with respect to time :

$$L = r^2\, \dot{\theta} \qquad (11)$$

This last expression is very well known, being described for instance by L. Landau and E. Lifchitz in their course “Mechanics”[9]. It shows that the areal velocity, defined as $\dot{f} = r^2 \dot{\theta}/2$, must be a constant as far as the angular momentum also is. Therefore the expression (11) is nothing else but the second law of Kepler.

Note that the time derivative of the true anomaly $\dot{\theta}$ and the frequency of rotation ω are related by the following formula :

$$\dot{\theta} = \omega\left(1 + e \cos\theta\right) \quad \text{or} \quad r\, \dot{\theta} = p\, \omega \qquad (12)$$

Now integrating the expression (11) over a complete period T of revolution for an ellipse (see L. Landau and E. Lifchitz[9]), knowing that the surface of an ellipse is $f = \pi\, a\, b$, a being the semi major axis and $b = \sqrt{a\, p}$ the minor one, we can integrate as so :

$$\int_0^T L dt = L\, T = \int_0^T 2\, \dot{f}\, dt = 2\, f = 2\, \pi\, a\, \sqrt{a\, p} \qquad (13)$$

Then using the equations (8), we are trivially led to the following formula :

$$L\, v_R = 4\, \pi^2\, \frac{a^3}{T^2} = k = \text{constant} \qquad (14)$$

This is the expression of the third law of Kepler.

In addition, we can see a deep connection between this Keplerian kinematics and the classical mechanics. Indeed by calculating the square of the velocity (1) it is trivial to get the following relationship :

$$\frac{1}{2}v^2-\frac{k}{r}=\frac{1}{2}v_R^2(e^2-1)=\text{constant} \qquad (15)$$

If we multiply this expression by the mass m of the orbiter, we get its classical mechanical energy[9], which is therefore :

$$E_M=\frac{1}{2}m\,v^2-\frac{mk}{r}=\frac{1}{2}m\,v_R^2(e^2-1) \qquad (16)$$

From this expression we see that the mechanical energy is minimum when the eccentricity is null, $E_M=-v_R^2/2$, and the energy is null when the eccentricity is equal to 1, i.e. when the trajectory is a parabola. Consequently, the energy necessary to extract an orbiter from the orbit of a central body, starting from a circular orbit, is simply $\Delta E=m\,v_R^2/2$.

Finally, we demonstrated that the expression (1) of the Keplerian velocity can predict the three laws of Kepler, as well as the mathematical expression of Newton's gravitational acceleration. It is then fully consistent with all what we know today about the Keplerian motion.

However the kinematics disagrees with Newton about the interpretation of his acceleration, it must be centripetal but not attractive, although the mathematical expression is the same in both cases. Concerning the planetary motions, Newton's acceleration is indeed kinematically fully compatible with a centripetal acceleration, however this is not true any more for the phenomenon of body falling.

In the following we will explore the consequence of this disagreement and propose a way to solve it. We will also propose a way to extend Newton's acceleration to other scales, by using the relation (5).

Body falling

For Newton the apple falling from the tree must experience an attractive force from the Earth, and then fall on a straight line, collinear to the attractive acceleration, as it must be. However we know that a straight line is not a limit, nor a particular case of the conic equation (8), as far as we must believe the laws of geometry.

Indeed a straight line cannot dependent upon an angle, so θ. Forcing θ to be a constant in the conic equation makes r constant, it can then only describe a point, not a line, and even less an accelerated straight trajectory. When the eccentricity is null, e=0, the conic trajectory is a circle, when e=1 it is a parabola, an ellipse between both, and when e>1 it is an hyperbola. None of these curves is a straight line. There are only 4 types of conics, and the straight line is not part of them.

A linear trajectory does not respect Kepler's first law that imposes a conic trajectory. It does not respect neither the second and third laws, as far as no surface nor areal velocity can be defined on a linear trajectory. Therefore, on this issue there is a conflict between Newton's postulate and Hamilton's kinematics. We have two ways solving it.

The first one consists to keep the conflict by assuming that the body falling is an other gravitational state, different from the general Keplerian state, and denying Hamilton's kinematics. If so, we must explain why and how a body can switch between these two different physical states. What are the physical criteria that make a body falling in a gravitational field, not following Kepler's laws, but rather an accelerated straight line. If we believe in Newton's attraction, this is a problem to solve.

The second one consists to accept the kinematics, and then we must explain why the apple is experimentally falling on a straight line. This can be achieved by considering the apple falling on a very sharp ellipse that could be confused locally with a straight line. For instance with an eccentricity equal to $e=1\text{-}10^{-20}$, the major axis of the ellipse being the Earth radius, the minor axis is not more than some tenth of a millimeter. Such an ellipse, which focus is at the Earth center of mass, and apogee at the altitude of the apple tree branch, is so sharp that it can be locally confused with a straight line. Let us explain the existence of such sharp ellipses.

When the apple stands on the tree, it has a null velocity, but must however respect the equation (1). The translation velocity must then be the exact opposite of the rotation velocity : $\mathbf{v_T} = -\mathbf{v_R}$. At a distance of one Earth radius, v_R is approximately 7.9 10^3m/s, and so huge is v_T , in the opposite direction. Exactly as $\mathbf{v_R}$ is the integral of the gravitational acceleration (5), $\mathbf{v_T}$ is the integral of all the other accelerations that are not gravitational, coming from "friction" forces. The apple would only suffer the gravitation, it would gravitate around the Earth on a circular orbit (v_R=7.9 10^3m/s, v_T=0m/s), but the tree blocks it, itself being blocked by the ground, and so on, each time increasing v_T.

When the apple disconnects from the tree, it is freed of a small part of these "friction" accelerations, so forces, that disabled its pure gravitational motion. Therefore v_T decreases a little bit, i.e. $v_T = (1 - \varepsilon)\, v_R$, where ε is very small. The overall Keplerian velocity (1) is then not null any more, the apple falls. But it falls on a conic which focus is at the Earth center of mass and which eccentricity is worth $e=v_T/v_R=1-\epsilon$. If ε is very small, such a conic is a very sharp ellipse, that could appear locally as a quasi straight line if the observer does not have measurement means that are precise enough. This is where we all confuse. The Keplerian ellipse of the falling apple is so sharp that we have no means to measure its curvature between the tree branch and the ground. Therefore we approximate it to a straight line.

The Keplerian kinematics explains that the Earth would be transparent, and all its mass concentrated in a single mathematical point, the apple would orbit around this point and get back to its initial position, exactly as any satellite does. And because the atoms are not mathematical points, but have a size, this process can also ensure that such thin ellipses are suitable to collide and aggregate many bodies, so to build planets and stars.

This is also the means used by the space agencies to return astronauts from space. From a circular satellited motion v_T=0, so purely gravitational, they decelerate, so they increase v_T, their trajectory becomes an ellipse which intersects the Earth sphere, so they can land. The more they break, the sharper the ellipse is. They would break further to have $v_T = (1 - \varepsilon)\, v_R$, ε being very small, they would look like the apple, falling straight forward to the ground, at a first approximation.

The kinematics thus tells us that what looks like an attraction is in reality the result of two antagonist main accelerations, the gravitational one, that makes the bodies naturally orbiting, and the non gravitational one, that slows down the orbiter. Therefore we should not describe the gravitation as the "universal attraction", but rather as the "universal rotation", because a totally free body in a gravitational field will have v_T=0 but never a null rotation velocity : $v_R = \sqrt{k/p} \neq 0$, where k is the Keplerian constant (14), and p the semi latus rectum of the conic.

Newton was then slightly wrong about the gravitation. He said "the moon is like the apple", but the kinematics rather says "the apple is like the moon". However this has no impact on our our day-to-day experience of the body falling. Its very sharp ellipse is so sharp that it can be indeed considered locally as a straight line at a first approximation.

So much more important is the mathematics of Newton's gravitational acceleration, that the kinematics agrees with, because it might bring the possibility to be used at other scales than the only astronomic one.

Universality of Newton's equation

The vector expression of Newton's acceleration is given by the following formula[9] :

$$\mathbf{a} = -\frac{GM}{r^3}\,\mathbf{r} \qquad (17)$$

where G is the universal constant of gravitation, and M the mass of the central body at the focus of the Keplerian conic. But for the kinematics this acceleration is given by the equation (5). Therefore to make both consistent, and remarking that GM and Lv_R have the same dimension in m^3s^{-2}, we shall accept :

$$L\, v_R = GM \qquad (18)$$

However on the left side of this equation we have a pure kinematic parameter demonstrated from the geometry and not submitted to any physical constraint. At the contrary what is on the right side is purely physical, and is stated as a postulate, the attraction. No doubt that this postulate is correct in our physical world, at a short astronomic scale, because we have widely verified it experimentally. But could it be only a particular case of the possible values that could adopt Lv_R at different scales ? Can we extend Newton's acceleration to some other scales, by considering Lv_R instead of GM, as the kinematics suggests with the relation (18) ? Let us take two examples of situations that goes in this direction.

Keplerian hydrogen

For the first example we note that Newton's acceleration has the same mathematical structure as Coulomb's acceleration[13], except that for an orbiting electron the constant $k_N = GM$ at the numerator is replaced by Coulomb's constant:

$$k_C = \frac{q^2}{4\,\pi\, m_e\, \epsilon_0} \qquad (19)$$

where q is the elementary charge, ε_0 is the permittivity of vacuum and m_e is the mass of the electron. Defined as so, k_N and K_C have the same dimension in m^3s^{-2}.

For the kinematics, if Lv_R=k_N at an astronomic scale, nothing forbids to have Lv_R=k_C at an atomic one. Therefore let us try to explain the property of the electron of the hydrogen atom from a Keplerian point of view. We can do so as far as the quantum mechanics allows both wave and particle interpretations for the electron.

If the electron is a Keplerian orbiter around the proton, it must be in weightlessness, therefore feeling no acceleration, and therefore emitting nothing. Actually orbiting on circles, such an electron fully respects the trajectories proposed by Bohr in his model[12].

We know from (16) that the energy required to extract the electron from the influence of the proton is $\Delta E = m_e v_R^2/2$. For the atom this energy is called "ionization energy", and must be equal to $\Delta E = \hbar\, \omega_I$, where ω_I is the ionization frequency, and $\hbar$ is Planck's constant divided by $2\,\pi$. Therefore we must verify:

$$v_R^2 = 2\,\frac{\hbar}{m_e}\,\omega_I \qquad (20)$$

In parallel the kinematics for a circular motion gives :

$$v_R^2 = L\,\omega \qquad (21)$$

Combining these two last expressions we get :

$$L = \frac{\hbar}{m} \ \text{ and } \ \omega = 2\,\omega_I \qquad (22)$$

Therefore knowing the numerical values of $\hbar$, m_e and ω_I , respectively 1.0546 10^{-34} kg m^2s^{-1}, 9.1095 10^{-31} kg and 2.071 10^{16} Hz, we can calculate v_R, 2.189 $10^6 ms^{-1}$, and also the constant k with the relation (14) $k = L\, v_R$. We get $k = 2.307\ 10^{-28}\ m^3 s^{-2}$ and this is the exact value of the above Coulomb's constant (19).

Knowing ω and v_R it is trivial to calculate the radius of the orbit, and we get $r = 0.529\ 10^{-10}$ m, which is Bohr's radius[12].

Now let us assume that the possible circular trajectories are characterized by the quantification of the fundamental angular momentum. The orbit n will then be characterized by $L_n = n L_1$, where n is an integer, and the consequent rotation velocity will be:

$$v_{Rn} = \frac{k}{L_n} = \frac{k}{n L_1} = \frac{v_{R1}}{n} \tag{23}$$

Regarding (16), the subsequent mechanical energy for an orbit n shall be:

$$E_n = -\frac{1}{2} m_e v_{Rn}^2 = -\frac{1}{2} m_e \frac{v_{R1}^2}{n^2} = \frac{E_1}{n^2} \tag{24}$$

The transition between two mechanical energy levels n_1 and n_2 will be given by :

$$\Delta E_{12} = E_1 \left(\frac{1}{n_2^2} - \frac{1}{n_1^2}\right) \tag{25}$$

This explains the emission/absorption spectrum of the hydrogen.

At last we shall note that the ratio between the electron velocity and the speed of light leads to write (with the above definition (19) of k_C):

$$\alpha = \frac{v_R}{c} = \frac{L\, v_R}{L\, c} = \frac{k_C}{L\, c} = \frac{q^2}{4 \pi \epsilon_0 \hbar c} \tag{26}$$

This is the expression of the fine structure constant as described by A. Sommerfeld[14].

Note that so far we did not have to suppose the existence of any electrical charge. Our way to calculate k_C does not require neither the electric charge, nor the permittivity of vacuum. We only applied the Keplerian kinematics to the electron of the hydrogen, assuming that it is in weightlessness on a circular orbit around the proton. Doing so, we considered that the electron is in gravitation around the proton with an other Keplerian constant k than the one assumed by Newton. Nevertheless we are able to calculate all its basic physical characteristics, even its spectrum, by the only knowledge of $\hbar$, m_e and its ionization energy ΔE_I. We do not pretend that such a particle description of the electron is the correct way to handle what is really an electron, especially regarding its wave properties, but just that regarding the electron like a particle, as allowed by the quantum mechanics, it looks like the Keplerian motion is deeply correlated to the electron properties in the atom.

Galaxy rotation

The second example is given by the problem of the galaxy rotation. Vera Rubin[11] measured that the stars in the disk of spiral galaxies have (approximately) the same velocity, whatever their distance to the galactic bulb, and this cannot be explained with Newton's acceleration. Indeed, at a first approximation we can consider that the stars in the galactic disk have a circular orbit given by the third law of Kepler (14) : $v = \sqrt{k/r}$. For Newton k=GM= constant, and consequently the velocity must decrease when the distance r increases. This is not what Vera Rubin measured, therefore, some postulated the existence of a dark matter, or either of a new kind of acceleration at long range, to be responsible for the observed behavior of the stars.

But the kinematics gives a simple solution to this problem. The constant k is worth $k = L\, v_R = L\, \omega\, r$, therefore:

$$v = \sqrt{L\, \omega} \tag{27}$$

and the velocity can remain constant whatever the distance, at the condition that $L\, \omega$ also is. In this case k must be proportional to r, but constant because r is constant for the circular motion that we consider at first approximation. Therefore the kinematics gives a solution to explain the observations, but this imposes a constraint : $L\, \omega$ is the same constant for all the stars of the disk. This can be verified experimentally. Note that $L\, \omega$ has the dimension of an energy divided by a mass, so this suggests that the stars of the disk are populating the same massless energy level $L\, \omega$. It then looks like a macroscopic and massless version of Planck-Einstein postulate[12] $E = \hbar\, \omega$, that would be at work in the structure of galaxies.

These two examples of interest, the Keplerian hydrogen and the galaxy rotation, are just proposals for investigation, they do not prove anything as such. They just show that using $k = Lv_R$, rather than restricting to k = GM, might make Newton's acceleration universal, rather than his only constant G, and consistent with the physical observations at different scales.

Conclusion

The kinematics that we presented here is trivial and agree all what we already know mathematically about the Keplerian motion. We made no use of any physical postulate, nor hypothesis, we just described the trivial kinematics deriving from the structure (1) of the Keplerian velocity demonstrated by Hamilton.

The derivation of Hamilton's velocity leads trivially to Kepler's laws as well as the mathematical structure of Newton's gravitational acceleration. However we shall notice two particular points.

First, for the kinematics the Keplerian acceleration is centripetal, but it is attractive for Newton. This has no impact on the planetary motions because in this case what Newton calls “attraction” is the acceleration that avoids the planet to remain on a straight line, so “attracting” it toward a center, and this is indeed a way to describe the centripetal acceleration. But taking the word “attraction” literally, causes a problem regarding the phenomenon of body falling. Because the trajectory of an attracted mobile must be collinear to the attractive acceleration, the apple must then fall from the tree on a straight line. But in this case the apple does not respect Kepler’s laws, despite all bodies should do by falling in a gravitational field. Fortunately the kinematics forecast that the trajectory of the apple must be a very thin ellipse, which focus is at the Earth center, so flattened that its curvature is negligible locally, and thus can be confused with a straight line at a first approximation.

Second, the kinematics suggests that Newton’s constant GM might be only a special value at a special scale of the broader kinematic factor $L\, v_R$ that has no physical constraint to respect. This is particularly questioning when considering the hydrogen’s electron like a Keplerian orbiter, orbiting in weightlessness on circular orbits around the proton. Indeed we show that all the kinematic characteristics of the electron as well as its spectrum, can be deduced from the Keplerian kinematics, without the knowledge of the existence of the electric charge. At the other scale size, the one of the galaxies, we show that Hamilton’s kinematics gives a possible mathematical solution to the rotations of spiral galaxies, with no need of any dark matter, nor of a modified Newtonian dynamics. All this suggest that, more than his constant G, what is universal in Newton’s acceleration is its intrinsic mathematical structure.

This could open some new perspectives, but a lot has still to be done to be affirmative on Newton’s gravitational acceleration being extendable to other scales. This article only outlines some tracks of thought about this issue.

Acknowledgment
Many thanks to Professor Émmanuel Trélat from Laboratoire Jacques-Louis Lions, for his help on the kinematic part of this article.

References

[1] W. R. Hamilton, The hodograph, or a new method of expressing in symbolic language the Newtonian law of attraction, Proc. R. Ir. Acad. III , 344353 (1845).

[2] David Derbes, Reinventing the wheel: Hodographic solutions to the Kepler problems. American Journal of Physics 69, 481 (2001).

[3] Orbit information derived from its hodograph, J. B. Eades, Tech. Rep. TM X-63301, NASA (1968)

[4] H. Abelson, A. diSessa and L. Rudolph, Velocity space and the geometry of planetary orbits, Am. J. Phys. 43 , 579-589 (1975).

[5] A. Gonzalez-Villanueva, H. N. Nunez-Yepez, and A. L. Salas-Brito, In veolcity space the Kepler orbits are circular, Eur. J. Phys. 17 , 168-171 (1996).

[6] T. A. Apostolatos, Hodograph: A useful geometrical tool for solving some difficult problems in dynamics, Am. J. Phys. 71 , 261-266 (2003).

[7] E. I. Butikov, The velocity hodograph for an arbitrary keplerian motion, Eur. J. Phys. 21 (2000) 1-10

[8] R. H. Battin, Astronautical guidance, McGraw-Hill, Electronic Science Series, New-York, 1964, §1.6, p 16.

[9] L.Landau, E. Lifchitz, Mechanics, Ed. Mir, Moscow, 1966, §15

[10] Douglas W. MacDougal, Newton's Gravity, Springer Science+Business Media New York, 2012, ISBN 978-1-4614-5443-4

[11] Rubin, V. C.; Ford Jr. W.K. & Thonnard, N., « Extended rotation curves of high-luminosity spiral galaxies. IV. Systematic dynamical properties, Sa -> Sc », The Astrophysical Journal, vol. 225, 1978, L107-L111

[12] Richard P. Feynman, Robert B. Leighton, Matthew Sands, The Feynman Lectures on Physics, vol. 3., Addison-Wesley, 1964

[13] Feynman, Richard P., The Feynman Lectures on Physics Vol II, (1970), ISBN 9780201021158.

[14]Arnold Sommerfeld, Zur Quantentheorie der Spektrallinien,Annalen der Physik, vol. 356, no 17, 1916, p. 1-94